CAN GENERAL-RELATIVISTIC DESCRIPTION OF GRAVITATION BE CONSIDERED COMPLETE?*

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The local galactic cluster, the Great attractor, embeds us in a dimensionless gravitational potential of about -3×10^{-5}. In the solar system this potential is constant to about 1 part in 10^{11}. Consequently, planetary orbits, which are determined by the gradient in the gravitational potential, remain unaffected. However, this is not so for the recently introduced flavor-oscillation clocks where the new redshift-inducing phases depend on the gravitational potential itself. On these grounds, and by studying the invariance properties of the gravitational phenomenon in the weak fields, we argue that there exists an element of incompleteness in the general-relativistic description of gravitation. An incompleteness-establishing inequality is derived and an experiment is outlined to test the thesis presented.

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I. The gradients of the gravitational potentials are well known to play a major role in the understanding of motion of the cosmic bodies. Especially in the weak-field limit of Einstein's theory of gravitation, they are responsible for the description of, say, the planetary orbits. In contrast to that, importantly, in the same limit, there are quantum mechanical effects that depend upon the gravitational potentials themselves. For example, it was recently shown that in performing a quantum mechanical linear superposition of different mass eigenstates of neutrinos belonging to different lepton generations, one may create a so-called "flavor oscillation clock" that has the remarkable property of redshifting precisely as required by the Einstein's theory of gravitation [1].

In the present study, we demonstrate that such clocks, in principle, allow to measure the essentially constant gravitational potential of the local clusters of the galaxies. Taken to its logical conclusion, this observation results in the question on the completeness of Einstein's theory of gravitation. In this essay we systematically explore this question. We come to the conclusion that, while the gravitationally induced accelerations vanish in a terrestrial free fall, the gravitationally induced phases of the flavor-oscillation clocks do not.

We begin with defining the context, then we derive an inequality on the incompleteness of the general-relativistic description of gravitation, this is followed by the outline of an experiment to test the derived inequality, and finally we make some concluding remarks and summarize the essential thesis of this essay.

II. As is well known, the solar system is embedded in the essentially constant gravitational potential of the local cluster of the galaxies, the so-called Great attractor. This gravitational potential, denoted by Φ_{GA} in the following, may be estimated over the entire solar system to be [2]

$$\text{Solar system:} \quad \Phi_{GA} \equiv \frac{1}{c^2}\,\phi_{GA} = -3\times 10^{-5}. \tag{1}$$

For this essay the precise value of Φ_{GA} is not important, but what is more relevant is that it is *constant* over the entire region of the solar system to an exceedingly large accuracy of 1 part in $R_{GA-S}/\Delta R_S$. Here ΔR_S represents the spatial extent of the solar system, and

R_{GA-S} is the distance of the solar system from the Great attractor. Taking ΔR_S to be of the order of Pluto's semi-major axis (i.e., approximately 40 AU), and R_{GA-S} to be about 40 Mpc [2], we obtain $R_{GA-S}/\Delta R_S \sim 10^{11}$. For comparison, the terrestrial and solar potentials on their respective surfaces are of the order $\Phi_E = -6.95 \times 10^{-10}$, $\Phi_S = -2.12 \times 10^{-6}$, and are therefore much smaller as compared to Φ_{GA}. Nonetheless, they carry significantly larger gradients over the relevant experimental regions.

Yet, the constant potential of the Great attractor that pervades the entire solar system is of no physical consequence within the general-relativistic context (apart from it being responsible for the overall local motion of our galaxy). Even the parenthetically observed motion disappears if we hypothetically and uniformly spread the matter of the galactic cluster into a spherical mass to concentrically surround the Earth. Such a massive shell in its interior provides an example of the gradientless contribution to the gravitational potential that we have in mind.

A terrestrial freely falling frame that measures accelerations to an accuracy of less than 1 part in about 10^{11} is completely insensitive to this constant potential. Similarly, since the planetary orbits are determined by the gradient of the gravitational potential, they too remain unaffected by this potential. Nonetheless, in what follows we shall show that quantum mechanical systems exist that are sensitive to Φ_{GA}. The simplest example for such a system is constructed in performing a linear superposition of, say, two different mass eigenstates (see Eqs. (9) and (10) below).

In the next section, Φ_{GA} shall be considered as a physical and gradientless gravitational potential as idealized in the example indicated above. This potential is to be distinguished from the usual "constant of integration" or the "potential at spatial infinity."

III. In the following we will exploit the weak-field limit of gravity as being introduced on experimental grounds. Here, the phrase "weak-field limit" refers to the experimentally established limit in the weak gravitational fields, rather than to the limit of a specific theory. Further, although not necessary, for the sake of the clarity of presentation we shall work in the non-relativistic domain and neglect any rotation that the gravitational source may have.

This assumption shall be implicit throughout this essay. The arguments shall be confined to the system composed of the Earth and the Great attractor, and are readily extendable to more general situations.

For the measurements on Earth the appropriate general-relativistic (GR) space-time metric is

$$ds^2 = g_{\mu\nu}^{GR} dx^\mu dx^\nu = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 + \frac{2GM}{c^2 r}\right) d\boldsymbol{r}^{\,2}, \tag{2}$$

where M is the mass of the Earth, r refers to the distance of the experimental region from Earth's center, and $d\boldsymbol{r}^{\,2} = (dx^2 + dy^2 + dz^2)$. The conceptual basis of the theory of general relativity asserts that the flat space-time metric $\eta_{\mu\nu}^{GR}$,

$$ds^2 = \eta_{\mu\nu}^{GR} dx^\mu dx^\nu = dt^2 - d\boldsymbol{r}^{\,2}, \tag{3}$$

is measured by a freely falling observer on Earth (or, wherever the observer is). In this framework, a stationary observer ($\mathcal{O}$) on the Earth may define a gravitational potential according to

$$\phi_E(\boldsymbol{r}) = \frac{c^2}{2}\left(g_{00}^{GR} - \eta_{00}^{GR}\right) = -\frac{c^2}{2}\left(g_{\jmath\jmath}^{GR} - \eta_{\jmath\jmath}^{GR}\right), \quad \jmath = 1, 2, 3 \text{ (no sum)}. \tag{4}$$

One immediately suspects that such a description may not incorporate the full physical effects of such physical potentials as ϕ_{GA} even though this conclusion is consistent with the classical wisdom. Indeed, the classical equation of motion consistent with the approximation in Eq. (2),

$$m_i \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\, m_g \boldsymbol{\nabla}\phi_E(\boldsymbol{r}), \tag{5}$$

is invariant under the transformation,

$$\phi(\boldsymbol{r})_E \rightarrow \varphi_E(\boldsymbol{r}) = \phi_{GA} + \phi_E(\boldsymbol{r}). \tag{6}$$

For this reason ϕ_{GA} has no apparent effect on the planetary orbits.

In the quantum realm the appropriate equation of motion is the Schrödinger equation with a gravitational interaction energy term,

$$\left[-\left(\frac{\hbar^2}{2m_i}\right)\boldsymbol{\nabla}^2+m_g\phi_{grav}(\boldsymbol{r})\right]\psi(t,\boldsymbol{r})=i\hbar\frac{\partial\psi(t,\boldsymbol{r})}{\partial t}, \tag{7}$$

as has been confirmed *experimentally* in the classic neutron interferometry experiments of Collela, Overhauser, and Werner [3,4]. Equation (7) is not invariant under the transformation of the type (6).

Moreover, this lack of invariance does not disappear in the relativistic regime where an appropriate relativistic wave equation, such as the Dirac equation, must be considered. Therefore, the gravitational potential that appears in Eq. (7) cannot be identified with $\phi_E(\boldsymbol{r})$ of Eq. (4). To treat the contributions from the Great attractor and the Earth on the same footing of physical reality, the following identification has to be made:

$$\phi_{grav}(\boldsymbol{r})\equiv\varphi_E(\boldsymbol{r})=\phi_{GA}+\phi_E(\boldsymbol{r}). \tag{8}$$

A second observation to be made is to note that while by setting $m_i=m_g$ in Eq. (5), the resulting equation becomes independent of the test-particle mass; this is *not* so for the quantum mechanical equation of motion (7) [4].

These two distinctions between the classical and quantum evolutions lead to the conclusion that the theory of general relativity for the description of gravitation cannot be considered complete. The gravitational potentials as defined via $g_{\mu\nu}(\boldsymbol{r})$ carry an independent physical significance in the quantum realm, a situation that is reminiscent on the significance of the gauge potential in electrodynamics as revealed by the Aharonov-Bohm effect [5].

The statement on the general-relativistic incompleteness is best illustrated on the example of a "flavor-oscillation clock" as introduced in Refs. [1,6]. Such clocks are constructed as a quantum mechanical linear superposition of different mass eigenstates (for instance, say, two neutrinos from two different lepton generations [7]),

$$|F_a\rangle=\cos(\theta)|m_1\rangle+\sin(\theta)|m_2\rangle, \tag{9}$$

$$|F_b\rangle=-\sin(\theta)|m_1\rangle+\cos(\theta)|m_2\rangle. \tag{10}$$

In the linear superposition of the mass eigenstates we assume (*only* for simplicity) that both $|m_1\rangle$ and $|m_2\rangle$ carry vanishingly small three momentum (i.e., are at rest).

By studying the time oscillation between the flavor states $|F_a\rangle$ and $|F_b\rangle$ one discovers that this system can be characterized by the flavor-oscillation frequency,

$$\Omega^{\infty}_{a\rightleftharpoons b} = \frac{(m_2 - m_1)\, c^2}{2\hbar}. \tag{11}$$

The superscript on $\Omega^{\infty}_{a\rightleftharpoons b}$ is to identify this frequency with a clock at spatial infinity from the gravitational sources under consideration (see below).

Now consider this flavor-oscillation clock to be immersed into the gravitational potential $\varphi_E(r)$. Then each of the mass eigenstates picks up a *different* phase because the gravitational interaction is of the form $m \times \varphi_E(r)$. As a result, one finds that the new flavor-oscillation frequency, denoted by $\Omega^{\mathcal{O}}_{a\rightleftharpoons b}$, is given by [6]

$$\Omega^{\mathcal{O}}_{a\rightleftharpoons b} = \left(1 + \frac{\varphi_E(\boldsymbol{r})}{c^2}\right) \Omega^{\infty}_{a\rightleftharpoons b}. \tag{12}$$

This equation is valid for an observer fixed in the global coordinate system attached to the Earth.

Equation (12) would have been the standard gravitational redshift expression if the $\varphi_E(\boldsymbol{r})$ was replaced by $\phi_E(\boldsymbol{r})$. Freely falling frames ($\mathcal{F}$) do not carry fastest moving clocks, they carry clocks that are sensitive to potentials of the type ϕ_{GA}. A freely falling frame in Earth's gravity only annuls the gradients of the gravitational potential while preserving all its constant pieces such as ϕ_{GA}. In denoting by $\Omega^{\mathcal{F}}_{a\rightleftharpoons b}$, the frequency as measured in a freely falling frame on Earth, one is led to

$$\Omega^{\mathcal{F}}_{a\rightleftharpoons b} = \left(1 + \frac{\phi_{GA}}{c^2}\right) \Omega^{\infty}_{a\rightleftharpoons b}. \tag{13}$$

From a physical point of view, ϕ_{GA} represents contributions from all cosmic-matter sources. However, all these contributions carry the same sign. In addition, in the context of the cosmos, $\Omega^{\infty}_{a\rightleftharpoons b}$ becomes a purely theoretical entity. Nevertheless, as shown below, $\Omega^{\infty}_{a\rightleftharpoons b}$ does have an operational meaning.

As a consequence, the following incompleteness-establishing inequality is found,

$$\Omega^{\mathcal{F}}_{a\rightleftharpoons b} < \Omega^{\infty}_{a\rightleftharpoons b}. \tag{14}$$

This is the primary result of our essay.

IV. To experimentally test the incompleteness of the general-relativistic description of gravitation and measure the essentially constant gravitational potential in the solar system, we rewrite Eqs. (12) and (13) into (to first order in the potentials)

$$\frac{\Omega^{\mathcal{O}}_{a\rightleftharpoons b}}{\Omega^{\mathcal{F}}_{a\rightleftharpoons b}} = 1 + \frac{\phi_E(\boldsymbol{r})}{c^2}, \tag{15}$$

$$\frac{\Omega^{\mathcal{O}}_{a\rightleftharpoons b}}{\Omega^{\infty}_{a\rightleftharpoons b}} = \frac{\phi_{GA}}{c^2} + \left(1 + \frac{\phi_E(\boldsymbol{r})}{c^2}\right). \tag{16}$$

Equation (15) shows how the ϕ_{GA}-dependence disappears in $\Omega^{\mathcal{O}}_{a\rightleftharpoons b}/\Omega^{\mathcal{F}}_{a\rightleftharpoons b}$. Equation (16), however, indicates that by systematically measuring $\Omega^{\mathcal{O}}_{a\rightleftharpoons b}$ as a function of $\boldsymbol{r}$, e.g., for an atomic system prepared as a linear superposition of different energy eigenstates, one can decipher the existence of ϕ_{GA}. Because all terrestrial clocks are influenced by the same ϕ_{GA}-dependent constant factor, it is essential that the flavor-oscillation clocks under consideration integrate the accumulated phase over different paths, thus probing different $\phi_E(\boldsymbol{r})$, and then return to the *same* spatial region in order that all the data interpretation refers to the same time standard. Such an integration is easily accommodated in Eq. (16). One would then make a *two-parameter fit* in $\{\Omega^{\infty}_{a\rightleftharpoons b}, \phi_{GA}\}$ to a large set of the *closed-loop integrated* data on $\{\Omega^{\mathcal{O}}_{a\rightleftharpoons b}(\boldsymbol{r}), \phi_E(\boldsymbol{r})\}$. Explicitly,

$$\oint_\Gamma \Omega^{\mathcal{O}}_{a\rightleftharpoons b}(\boldsymbol{r}) d\ell(\boldsymbol{r}) = \Omega^{\infty}_{a\rightleftharpoons b}\left(1 + \frac{\phi_{GA}}{c^2}\right)\oint_\Gamma d\ell(\boldsymbol{r}) + \frac{\Omega^{\infty}_{a\rightleftharpoons b}}{c^2}\oint_\Gamma \phi_E(\boldsymbol{r}) d\ell(\boldsymbol{r}), \tag{17}$$

where $d\ell(\boldsymbol{r})$ is the differential length element along the closed path Γ. By collecting the data on the "accumulated phase" $\oint_\Gamma \Omega^{\mathcal{O}}_{a\rightleftharpoons b}(\boldsymbol{r}) d\ell(\boldsymbol{r})$ and the "probed gravitational potential" $\oint_\Gamma \phi_E(\boldsymbol{r}) d\ell(\boldsymbol{r})$ for a set of Γ, and fitting a straight line, one may extract $\{\Omega^{\infty}_{a\rightleftharpoons b}, \phi_{GA}\}$. Rigorously speaking, what one obtains is $\Omega^{\infty}_{a\rightleftharpoons b}$ and the constant ϕ_{GA} as modified by other cosmic contributions. Further, these additional contributions may include extra general-relativistic

contributions from the yet-unknown interactions that may couple to the various parameters associated with the superimposed quantum states.

A simple consideration of the magnitude of various gravitational potentials involved and the accuracy of clocks based on quantum superpositions of atomic states leads to the tentative conclusion that the suggested experiment is feasible within the existing technology. In this regard note is taken that various ionic and atomic clocks have reached an accuracy of 1 part in 10^{15} with a remarkable long-term stability. In addition, workers in this field are optimistic that a several-orders-of-magnitude improvement may be expected in the next few years (see, e.g, Barbara Levi's recent coverage of this subject in the February 1998 issue of *Physics Today* [8]).

V. In the present study we emphasized observability of the constant potential of the Great attractor by means of flavor-oscillation clocks. While in a classical context, the force $\boldsymbol{F} = -m_g \boldsymbol{\nabla}\phi(\boldsymbol{r})$ experienced by an object is independent of gradientless gravitational potentials such as ϕ_{GA}; the frequency of the flavor oscillation clocks depends directly on ϕ_{GA} [in addition to $\phi_E(\boldsymbol{r})$].

The above considerations suggest that in a free fall the space-time interval is given by

$$\mathcal{F}: \quad ds^2_{\mathcal{F}} = \eta^{\mathcal{F}}_{\mu\nu} dx^\mu dx^\nu = \left(1 + \frac{2\phi_{GA}}{c^2}\right) dt^2 - \left(1 - \frac{2\phi_{GA}}{c^2}\right) d\boldsymbol{r}^{\,2}, \tag{18}$$

and not by Eq. (3), as asserted by the foundations of the theory of general relativity. Simultaneously, Eq. (2) is to be replaced by

$$\mathcal{O}: \quad ds^2_{\mathcal{O}} = g^{\mathcal{O}}_{\mu\nu} dx^\mu dx^\nu = \left(1 + \frac{2\varphi_E(\boldsymbol{r})}{c^2}\right) dt^2 - \left(1 - \frac{2\varphi_E(\boldsymbol{r})}{c^2}\right) d\boldsymbol{r}^{\,2}, \tag{19}$$

with Eq. (3) now remaining valid only at the "spatial infinity"

$$\infty: \quad ds^2_\infty = \eta^\infty_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\boldsymbol{r}^{\,2}. \tag{20}$$

The symbols $\{\mathcal{F}:,\ \mathcal{O}:,\ \infty:\}$ in the above equations are to remind the reader of the related observers. These equations are expected to hold at least in the quantum realm.

Such modifications are perfectly justified because of the linearity of the weak-field limit, where one is able to formulate the physics in terms of the additive gravitational potentials.

Within the considered framework and approximations, the space-time curvatures derived from $g_{\mu\nu}^{GR}$ and $g_{\mu\nu}^{\mathcal{O}}$ are identical. A similar statement applies to $\eta_{\mu\nu}^{GR}$ and $\eta_{\mu\nu}^{\mathcal{F}}$. Yet, the quantum effects of gravitation do not vanish in a freely falling frame, they vanish at spatial infinity. Consequently, the observable gravitational potential (as detected, e.g., by the flavor-oscillation clocks) is given by

$$\phi_{grav}(\boldsymbol{r}) = \frac{c^2}{2}\left(g_{00}^{\mathcal{O}}(\boldsymbol{r}) - \eta_{00}^{\infty}\right). \tag{21}$$

This result is in agreement with Eq. (8) and differs from the general-relativistic result contained in Eq. (4). The theory of general relativity implicitly assumes equality of $\eta_{\mu\nu}^{\infty}$ and $\eta_{\mu\nu}^{\mathcal{F}}$, and thereby omits physical effects of the gradientless physical potentials in its treatment of the freely falling observers. Here, by examining the classical and quantum realm in the weak gravitational fields (where the difficulties of "quantum gravity" are avoided), we have shown that this implicit general-relativistic assumption has to be abandoned, because while $\eta_{\mu\nu}^{\infty} = \eta_{\mu\nu}^{GR}$, $\eta_{\mu\nu}^{\mathcal{F}} \neq \eta_{\mu\nu}^{GR}$. That is, $\eta_{\mu\nu}^{\mathcal{F}}$ is to be physically distinguished from $\eta_{\mu\nu}^{\infty}$.

The reported incompleteness in the theory of general relativity for the description of gravitation also reveals certain similarities to the Aharonov-Bohm effect [5]. Indeed, in the Aharonov-Bohm effect an observable phase arises in a region with vanishing field strength tensor $F^{\mu\nu}(\boldsymbol{r})$, (i.e., in a region with vanishing 4-curl of the gauge potential $A^{\mu}(\boldsymbol{r})$). In the effect reported here, an observable phase arises in a region where the contributions of the ϕ_{GA}-type constant potentials to the curvature tensor $R^{\mu\nu\sigma\lambda}(\boldsymbol{r})$ vanish. Both of the effects mentioned above illustrate the circumstance that in quantum mechanical processes the gauge potential $A^{\mu}(\boldsymbol{r})$ and the gravitational potential $g^{\mu\nu}(\boldsymbol{r})$ may be favored over the corresponding fields strength tensor $F^{\mu\nu}(\boldsymbol{r})$, and the curvature tensor $R^{\mu\nu\sigma\lambda}(\boldsymbol{r})$, respectively.

However, since the number of the independent degrees of freedom of $A^{\mu}(\boldsymbol{r})$ is quite different from that of $g^{\mu\nu}(\boldsymbol{r})$, the analogy between the Aharonov-Bohm effect and the one considered here is not complete.

In summary, the local galactic cluster, the Great attractor, embeds us in a dimensionless gravitational potential of about -3×10^{-5}. In the solar system this potential is constant

to about 1 part in 10^{11}. Consequently, planetary orbits remain unaffected. However, this is not so for the flavor-oscillation clocks. In a terrestrial free fall the gravitationally induced accelerations vanish, but the gravitationally induced phases of the flavor-oscillation clocks do not. We argue that there exists an element of incompleteness in the general-relativistic description of gravitation. The arrived incompleteness may be subjected to an experimental test by verifying the inequality derived here.

The origin of the reported incompleteness lies in the implicit general-relativistic assumption on the equivalence of the space-time metric as measured by a freely falling observer in the vicinity of a gravitating source (which in turn is embedded in a Φ_{GA}-type constant gravitational potential) and the space-time metric as measured by an observer at "spatial infinity."

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I feel obliged to record a suggestion of Sam Werner. In a conversation in Missouri, about a year ago, he brought to my attention that he wishes to surround one of the arms of a neutron interferometer in a hollow cylinder filled with about a ton (if I remember correctly) of mercury. Thus, effects of a constant gravitational potential could be experimentally studied in a neutron interferometer. It appears that by appropriately varying the amount of mercury in the hollow cylinder one may also address the incompleteness issue of the theory of general relativity. It is also my pleasure to thank V. Raatriswapan for many useful comments. This work was done, in part, under the auspices of the U.S. Department of Energy.

REFERENCES

[1] Ahluwalia, D. V. and Burgard, C. (1998). *Phys. Rev.* D**57** (8).

[2] Kenyon, I. R. (1990). *Phys. Lett.* B**237**, 274.

[3] Collela, R., Overhauser, A. W., and Werner, S. A. (1975). *Phys. Rev. Lett.* **34**, 1472.

[4] Sakurai, J. J. (1985). *Modern Quantum Mechanics* (Benjamin/Cummings, Menlo Park California,), pp. 126–129.

[5] Aharonov, Y. and Bohm, D. (1959). *Phys. Rev.* **115**, 485.

[6] Ahluwalia, D. V. (1997). *Gen. Rel. Grav.* **29**, 1491.

[7] Ahluwalia, D. V. and Burgard, C. (1996). *Gen. Rel. Grav.* **28**, 1161; *Errata* (1997). **29**, 681.

[8] Levi, B. G. (1998). *Phys. Today* **51** (2), 21.